Problem 25 in Section 16.3

Recall that in this problem we are studying the pyramid bounded by the planes $z = -6$, $y = 0$, $y - x = 4$ and $2x + y + z = 4$. In class we calculated all the vertices of this pyramid. Now I will calculate them in Mathematica.

In[1]:= eqs = \{z = -6, y = 0, y - x = 4, 2x + y + z = 4\}

Out[1]= \{z = -6, y = 0, -x + y = 4, 2x + y + z = 4\}

Each vertex is at the intersection of three planes. So, I solve three of the four equations and get four vertices.

In[2]:= pP1 = \{x, y, z\} /. Solve[\{z = -6, y = 0, y - x = 4\}, \{x, y, z\}]\[1]

Out[2]= \{-4, 0, -6\}

In[3]:= pP2 = \{x, y, z\} /. Solve[\{z = -6, y = 0, 2x + y + z = 4\}, \{x, y, z\}]\[1]

Out[3]= \{5, 0, -6\}

In[4]:= pP3 = \{x, y, z\} /. Solve[\{z = -6, y - x = 4, 2x + y + z = 4\}, \{x, y, z\}]\[1]

Out[4]= \{2, 6, -6\}

In[5]:= pP4 = \{x, y, z\} /. Solve[\{y = 0, y - x = 4, 2x + y + z = 4\}, \{x, y, z\}]\[1]

Out[5]= \{-4, 0, 12\}

Plot these points:
There are 6 lines of interest

Cyan

In[7]:= \(\text{liC} = \{x, y, z\} / \text{Solve}[\{z = -6, y = 0\}, \{x, y, z\}]\) \[1\]

Solve::svars: Equations may not give solutions for all "solve" variables. \[\]

Out[7]= \(\{x, 0, -6\}\)

Green

In[8]:= \(\text{liG} = \{x, y, z\} / \text{Solve}[\{z = -6, y - x = 4\}, \{x, y, z\}]\) \[1\]

Solve::svars: Equations may not give solutions for all "solve" variables. \[\]

Out[8]= \(\{x, 4 + x, -6\}\)

Blue
In[9] = liB = {x, y, z} /. Solve[{z == -6, 2 x + y + z == 4}, {x, y, z}][[1]]

Solve::svars : Equations may not give solutions for all "solve" variables. >>

Out[9] = {x, 10 - 2 x, -6}

Red

In[10] = liR = {x, y, z} /. Solve[{y == 0, y - x == 4}, {x, y, z}][[1]]

Solve::svars : Equations may not give solutions for all "solve" variables. >>

Out[10] = {-4, 0, z}

Magenta

In[11] = liM = {x, y, z} /. Solve[{y == 0, 2 x + y + z == 4}, {x, y, z}][[1]]

Solve::svars : Equations may not give solutions for all "solve" variables. >>

Out[11] = {x, 0, 4 - 2 x}

Yellow

In[12] = liY = {x, y, z} /. Solve[{y - x == 4, 2 x + y + z == 4}, {x, y, z}][[1]]

Solve::svars : Equations may not give solutions for all "solve" variables. >>

Out[12] = {x, 4 + x, -3 x}

■ How to integrate? (easier way)

How to integrate? Fix y to be y0 and find the corresponding points on Green, Blue and Yellow lines

In[13] = liG

Out[13] = {x, 4 + x, -6}

In[14] = liB

Out[14] = {x, 10 - 2 x, -6}

In[15] = liY

Out[15] = {x, 4 + x, -3 x}

In[16] = liG /. Solve[liG[[2]] == y0, x] [[1]]

Out[16] = {-4 + y0, y0, -6}

In[17] = liB /. Solve[liB[[2]] == y0, x] [[1]]

Out[17] = {10 - y0, 1/2 y0, -6}

In[18] = liY /. Solve[liY[[2]] == y0, x] [[1]]

Out[18] = {-4 + y0, y0, -3 (-4 + y0)}
\[ y_0 = 2.5; \]
\[ p_{PG} = \{-4 + y_0, y_0, -6\}; \]
\[ p_{PB} = \{-4 + y_0, y_0, -3\}; \]
\[ p_{PY} = \{-4 + y_0, y_0, -6\}; \]
\[ \text{Graphics3D[} \]
\[ \quad \{\text{PointSize[0.02], Point[pP1], Point[pP2], Point[pP3], Point[pP4]}, \]
\[ \quad \{\text{Opacity[0.3], Polygon[{pP1, pP2, pP3]}, \]
\[ \quad \quad \quad \text{Polygon[{{pP1, pP2, pP4}], Polygon[{{pP1, pP4, pP3}], Polygon[{{pP4, pP2, pP3}]}}, \]
\[ \quad \quad \quad \text{Thickness[0.01], Cyan, Line[{{pP1, pP2}], Green, Line[{{pP1, pP3}], Blue, Line[{{pP2, pP3}], \]
\[ \quad \quad \quad \text{Red, Line[{{pP1, pP4}], Magenta, Line[{{pP2, pP4}], Yellow, Line[{{pP3, pP4}]}, \]
\[ \quad \quad \quad \text{Text[P1, pP1, \{1, 1\}], Text[P2, pP2, \{-1, 1\}], \]
\[ \quad \quad \quad \text{Text[P3, pP3, \{-1, -1\}], Text[P4, pP4, \{-1, -1\}]}], \]
\[ \quad \text{Polygon[{{pPG, pPB, pPY}]}] \}
\[ \}], \]
\[ \text{PlotRange -> \{-5, 6\}, \{-1, 8\}, \{-7, 13\}], \]
\[ \text{Axes -> True, BoxRatios -> \{11, 9, 20\}, AxesLabel -> \{x, y, z\}}\]
\]
In[21]:= Clear[y0]; Manipulate[pPG = \{\(-4 + y0, y0, -6\)\};

\[\text{pPB} = \left\{\frac{10 - y0}{2}, y0, -6\right\}; \text{pPY} = \{-4 + y0, y0, -3 (-4 + y0)\}\;];

Graphics3D[
{PointSize[0.02], Point[pP1], Point[pP2], Point[pP3], Point[pP4]},
{Opacity[0.3], Polygon[{pP1, pP2, pP3}],
 Polygon[{pP1, pP2, pP4}], Polygon[{pP1, pP4, pP3}], Polygon[{pP4, pP2, pP3}]},
{Thickness[0.01], Cyan, Line[{pP1, pP2}], Green, Line[{pP1, pP3}],
 Blue, Line[{pP2, pP3}], Red, Line[{pP1, pP4}], Magenta,
 Line[{pP2, pP4}], Yellow, Line[{pP3, pP4}]], {Text[P_1, pP1, \{1, 1\}],
 Text[P_2, pP2, \{-1, 1\}], Text[P_3, pP3, \{-1, -1\}], Text[P_4, pP4, \{-1, -1\}]}},
PlotRange \rightarrow \{-5, 6\}, \{-1, 8\}, \{-7, 13\}],
Axes \rightarrow True, BoxRatios \rightarrow \{11, 9, 20\}, AxesLabel \rightarrow \{x, y, z\}],
{(y0, 2), 0, 6}]
In[22]:= Integrate[Integrate[Integrate[1, \{z, -6, 4 - 2 x - y\}], \{x, y - 4, \frac{10 - y}{2}\}], \{y, 0, 6\}]
Out[22]= 162

- How to integrate? (harder way)

How to integrate? Fix z to be \(z_0\) and find the corresponding points on Red, Magenta and Yellow lines:

In[23]:= liR
Out[23]= \{-4, 0, z\}

In[24]:= liM
Out[24]= \{x, 0, 4 - 2 x\}

In[25]:= liY
Out[25]= \{x, 4 + x, -3 x\}

In[26]:= liR /. Solve[liR[3] == z0, z][1]
Out[26]= \{-4, 0, z0\}

In[27]:= liM /. Solve[liM[3] == z0, x][1]
Out[27]= \{\frac{4 - z0}{2}, 0, z0\}

In[28]:= liY /. Solve[liY[3] == z0, x][1]
Out[28]= \{-\frac{z0}{3}, 4 - \frac{z0}{3}, z0\}
Clear[z0]; Manipulate[
  pPR = {-4, 0, z0}; pPM = \left\{ \frac{4 - z0}{2}, 0, z0 \right\}; pPY1 = \left\{ \frac{z0}{3}, 4 - \frac{z0}{3}, z0 \right\};

  Graphics3D[
    {PointSize[0.02], Point[pP1], Point[pP2], Point[pP3], Point[pP4]},
    {Opacity[0.3], Polygon[{pP1, pP2, pP3}],
      Polygon[{pP1, pP2, pP4}],
      Polygon[{pP1, pP4, pP3}],
      Polygon[{pP4, pP2, pP3}]
    },
    {Thickness[0.01], Cyan, Line[{pP1, pP2}], Green, Line[{pP1, pP3}],
      Blue, Line[{pP2, pP3}], Red, Line[{pP1, pP4}], Magenta,
      Line[{pP2, pP4}], Yellow, Line[{pP3, pP4}]
    },
    {Thickness[0.01], Cyan, Line[{pPR, pPM}], Green,
      Line[{pPR, pPY1}], Blue, Line[{pPM, pPY1}]
    },
    {PointSize[0.02], Point[pPR], Point[pPM], Point[pPY1]},
    {Polygon[{pPR, pPM, pPY1}]}
  ],
  PlotRange -> {{-5, 6}, {-1, 8}, {-7, 13}},
  Axes -> True, BoxRatios -> {11, 9, 20}, AxesLabel -> \{x, y, z\},
  {{z0, 2}, -6, 12}]

Out[29]=
Recall

\[ \text{In[30]}:= \text{pPY1} = \left\{ \frac{-z_0}{3}, \frac{4 - z_0}{3}, z_0 \right\} \]

\[ \text{Out[30]}= \left\{ \frac{-z_0}{3}, \frac{4 - z_0}{3}, z_0 \right\} \]

So, \( y \) is between 0 and \( 4 - \frac{z}{3} \).

Recall, the green and blue line, but now change them to be at the level \( z_0 \)

Green

\[ \text{In[31]}:= \text{liGz} = \{x, y, z\} \/. \text{Solve}\left[\{z = z_0, y - x = 4\}, \{x, y, z\}\right] \]

Solve::svars : Equations may not give solutions for all "solve" variables. \( \triangleright \)

\[ \text{Out[31]}= \{x, 4 + x, z_0\} \]

Blue

\[ \text{In[32]}:= \text{liBz} = \{x, y, z\} \/. \text{Solve}\left[\{z = z_0, 2x + y + z = 4\}, \{x, y, z\}\right] \]

Solve::svars : Equations may not give solutions for all "solve" variables. \( \triangleright \)

\[ \text{Out[32]}= \{x, 4 - 2x - z_0, z_0\} \]

\[ \text{In[33]}= \text{Integrate[Integrate[Integrate[1, \{x, y - 4, \frac{4 - y - z}{2}\}], \{y, 4 - \frac{z}{3}\}], \{z, -6, 12\}] } \]

\[ \text{Out[33]}= 162 \]